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# Natural convection in a long vertical cylinder under gravity modulation

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This study is devoted to the onset of convection in differentially heated cylinders under gravity modulation. It specifically concerns the case of a vertical cylinder of infinite length, when a negative temperature gradient is maintained in the upward direction. The effect of modulation on the stability limits given by linear theory in the standard steady case is analysed. A method based on Floquet theory is proposed in the case of small values of the modulation amplitude $\epsilon$, for a fixed value of the frequency $\omega$. A general technique, called matrix method, which can easily be adapted to various kinds of geometries and boundary conditions, has been developed. Analytical approaches have been derived in some cases. Finally, an asymptotic analysis is presented for large $\omega$, under very general boundary conditions and periodic constraints, for finite $\epsilon$. An asymptotic relation is established for the onset of convection under periodic gravity modulation for large $\omega$ values, when $\epsilon \ll \omega$; the mathematical and physical foundations of this inequality are discussed.

## 1. Introduction

A lot of scientific experiments dealing with materials processing or fundamental fluid physics have been performed for several years by using the low level of gravitation aboard an orbiting spacecraft. Compared to $1g$ on the ground the residual microgravity ($\mu g$-) level still affects physical processes; in addition, this $\mu g$-vector field is characterized by a steady-state component and by a fluctuating contribution called $g$-jitter. Both these steady-state and fluctuating components can affect fluid phases subjected to thermal (or solutal) gradients. Recent studies have been devoted to measuring these components during the D1-Spacelab mission, by Hamacher & Merbold (1985) and Hamacher, Merbold & Jilg (1986*a*, *b*). These authors used two accelerometer systems. One was operating in a peak detection mode (1 Hz) to reduce the total amount of data (peak values within intervals of 1 s were detected from an analogue random response in positive and negative directions of the coordinate considered); it allows one to analyse a band width up to 100 Hz. The second operated in a high-rate sampling mode, suitable for performing frequency analysis up to about 5 Hz.

We are directly involved in collaborative work with groups doing experiments about materials science and fluid physics in space. Most of these experiments are done in cylindrical furnaces, as in the D1-mission (Billia *et al.* 1987; Camel *et al.* 1987; Henry & Roux 1987). Several other experiments are now prepared. A complete understanding of the role of the gravity modulation on the convective motions in fluids is needed to better analyse the available results and to better design future experiments.

According to previous workers (Hamacher *et al.* 1986; Malméjac *et al.* 1981;

| | Kinematic viscosity $\nu$ ($cm^2\,s^{-1}$) | Thermal diffusivity $\kappa$ ($cm^2\,s^{-1}$) | Prandtl number $Pr$ |
|---|---|---|---|
| Gases | $10^{-1}$ | $10^{-1}$ | 1 |
| Liquid metals | $10^{-3}$–$10^{-1}$ | $10^{-2}$–$10^{0}$ | $10^{-3}$–$10^{-1}$ |
| Organic liquids | $10^{-3}$–$10^{-2}$ | $10^{-4}$–$10^{-3}$ | $10^{-3}$–$10^{-1}$ |
| Molten glasses | $10^{2}$ | $10^{-2}$ | $10^{3}$–$10^{4}$ |
| Molten salts | $10^{-3}$–$10^{-2}$ | $10^{-3}$ | $10^{0}$–$10^{1}$ |
| Silicone oils | $10^{-2}$–$10^{1}$ | $10^{-4}$ | $10^{1}$–$10^{4}$ |
| Water | $10^{-2}$ | $10^{-3}$ | $10^{1}$ |

TABLE 1. Typical physical fluid properties

Ostrach 1976, 1982), two types of frequencies have to be considered. One type corresponds to attitude changes during the orbital motion (very small frequencies); the other one ($g$-jitter), which corresponds to spacecraft manoeuvres and mechanical vibrations, gives rise to a random frequency band varying from 0.1 to 100 Hz.

Most of the experiments to be considered are performed in long cylindrical containers of small radius (typically one or a few cm) and in furnaces delivering axial temperature gradient. Different materials are used, corresponding to fluids covering a wide range of Prandtl numbers (see table 1).

The aim of this study is to analyse the effect of gravity modulation on the onset of convection in differentially heated cylinders, considering a wide range of values of $\omega$ and $Pr$. As a first step, the study is limited to the case of vertical cylinders of infinite length, when a negative temperature gradient is maintained in the upward direction. After giving the governing equations for small perturbations in §3, and the formulation of the stability problem, by using the Floquet theory and expansions of the variable in terms of Bessel functions in §4, we propose two techniques to solve these perturbation equations. The first one, which is analytical, is presented in §5 for boundary conditions of adiabatic type. The second, called matrix method, is presented for more general boundary conditions in §6. An asymptotic analysis has been considered for large frequencies; two methods are proposed, respectively, in §5.2 as an application of the analytical method for small-amplitude modulations, and in §7 through an original approach, valid for finite-amplitude modulations. Comparisons with the available results in the literature concerning the alteration of the stability threshold due to different kinds of modulation (gravity or surface temperature) will be made, although most of these available results concern the classical Bénard problem (i.e. without sidewall confinement).

## 2. Presentation of the problem

As reported by Ostrach (1976), it was found by previous authors (Richardson 1967; Pak, Winter & Schoenals 1970; Gershuni, Zhukhovitskii & Iurkov 1970) that vibrations can either substantially enhance or retard heat transfer and drastically affect convection, by altering the transitions from quiescent to laminar flow (critical Rayleigh number) and from laminar to turbulent flow.

A lot of works have been devoted to the effect of unsteady constraints on the onset of convective motion. The mathematical difficulty lies in the fact that the equations describing the growth of initial disturbances are non-autonomous and that the method of normal modes is not applicable. Several investigations have been made in the case of periodic constraints applied to different shear flows or buoyancy driven motions.

Many authors have established the significant influence of a time-periodic excitation on the stability threshold. Hall (1975) studied the linear and nonlinear stability of the modulated Couette flow by means of the small-parameter method. In addition, he considered the case of large frequencies and arbitrary amplitude. The modulated Poiseuille flow has been studied by Grosch & Salwen (1968) who used a fourth-order Runge–Kutta method (as proposed by Ralston 1962) for the numerical integration of a system of ($N$ first-order) differential equations.

The problem of the convective stability in the presence of a periodically varying parameter has first been emphasized by Gershuni & Zhukhovitskii (1963), who established the alteration of the stability threshold, by using the first-order approximation (one trial function) of Galerkin's method. This method, which reduces a first-order differential system to an ordinary differential equation with periodic coefficients, permits one to handle the problem with an arbitrary modulation amplitude. It has been used for the classical (unconfined) Bénard problem and for a flow confined in a vertical cylinder heated from the bottom (as the one considered herein) by Gershuni *et al.* (1970). For large frequencies, these authors established an analytical relation that will be discussed in §8.

Most of the other studies devoted to the modulation of the surface temperature or the gravity, in buoyancy driven flows, concern only the (unconfined) Bénard problem with different kinds of boundary conditions. In the case of small amplitude modulations, a basic linear stability analysis has been done by Venezian (1969), who studied the effect of the modulation of the temperature gradient on two-dimensional small disturbances. Roppo, Davis & Rosenblat (1984) also studied the effect of the same kind of modulation, but on three-dimensional disturbances; in addition, they considered the nonlinear stability analysis. For finite amplitude modulations, in addition to Gershuni *et al.* (1970) who used only one trial function (that corresponds, in fact, to a separation of variables solution), several authors (Gresho & Sani 1970; Rosenblat & Herbert 1970; Rosenblat & Tanaka 1971; Yih & Li 1972) used a Galerkin technique with a small set of trial functions. This small number appeared sufficient in the case of rigid and conducting horizontal walls considered by these authors, who all establish in different manners the alteration of the stability limit under the modulation of constraints.

The great difficulty lies in the choice of the stability criterion, as discussed by Homsy (1974); this paper is not dealing with this problem, neither with the other fundamental question concerning the ability of the linear theory to predict instability. The use of Floquet theory in this paper is based on the stability of the null solution in the sense of Liapunov, and thus implies the prediction of the asymptotic stability. In the simpler case, when Mathieu's equation gives a good approximation to the equation of perturbation, one has a simple criterion for linear stability. But, in general, Mathieu's equation does not describe all the properties of evolution systems of the Navier–Stokes type. The Floquet theory is in these cases the best approach to the problem. In other cases, the energy method gives sufficient conditions for stability, as shown by Homsy (1974) for instance.

## 3. Governing equations

We consider specifically the case of gravity modulation,

$$g = g_0 + \epsilon^* \cos \omega^* t^*, \tag{3.1}$$

where $\epsilon^*$, $\omega^*$ and $t^*$ represent, respectively, dimensional amplitude, frequency and time, and $g_0$ the mean gravity (for space application, ($g_0$ could be $10^{-3}$ m s$^{-2}$ or less) applied to a column of a viscous incompressible fluid, in the form of a cylinder of infinite length, subjected to a constant temperature gradient,

$$\mathrm{d}T/\mathrm{d}z = -\gamma. \tag{3.2}$$

All fluid properties are constant, except that the density, $\rho$, is varying linearly with the temperature in the buoyancy terms according to the Boussinesq approximation:

$$\rho = \rho_0[1-\alpha(T-T_0)],$$

where $\alpha$ is the volume expansion coefficient and the subscript $\theta$ represents the mean condition. Also, we neglect the viscous dissipation terms in the energy equation. Thus, following Bird, Stewart & Lightfoot (1960, p. 388) the governing equations read:

$$-\mathrm{D}\boldsymbol{U}/\mathrm{D}t + \nu\nabla^2 \boldsymbol{U} + [1-\alpha(T-T_0)]\, g\boldsymbol{k} - \rho_0^{-1}\nabla P = 0, \tag{3.3a}$$

$$\nabla \cdot \boldsymbol{U} = 0, \tag{3.3b}$$

$$-\mathrm{D}T/\mathrm{D}t + \kappa\nabla^2 T = 0. \tag{3.3c}$$

In the above, $\boldsymbol{U}$, $P$, $\nu$ and $\kappa$ are, respectively, the velocity, the pressure, the kinematic viscosity and the thermal diffusivity; $\boldsymbol{k}$ represents the unit vector upward, in the positive $z$-direction (antiparallel to gravity); D represents the substantive derivative, and $\nabla^2$ is the Laplacian of a vector field in equation (3.3$a$) and of a scalar one in equation (3.3$c$), the expressions for which in cylindrical coordinates are given by Bird *et al.* (1960). (see also Charlson & Sani 1970.)

These equations admit an equilibrium solution in which $\boldsymbol{U} = 0$, $T = T(z,t)$ is a solution of

$$-\partial T/\partial t + \kappa\nabla^2 T = 0,$$

and the pressure $p(z,t)$ balances the buoyancy forces. Of course, the precise form of $T(z,t)$ depends on the boundary conditions.

In the equations (3.3) the following splitting of the variables can be used:

$$T = T(z,t) + \theta'(r,\phi,z,t), \quad P = p(z,t) + p'(r,\phi,z,t),$$

where $\theta'$ and $p'$ represent (small) perturbations of the temperature and the pressure due to the convective motion. After linearization of the equations (3.3), we obtain the usual form of the small-perturbation equations (see for example Gershuni & Zhukhovitskii 1963). These equations, for the velocity $\boldsymbol{U}$ of the components $(u, v, w)$ and for $\theta$ and $p$, in non-dimensional form, can be written in cylindrical coordinates $(r, \phi, z)$ as

$$-Pr^{-1}\partial \boldsymbol{U}/\partial t + \nabla^2 \boldsymbol{U} + R(t)\,\theta \boldsymbol{k} - \nabla p = 0, \tag{3.4a}$$

$$\nabla \cdot \boldsymbol{U} = 0, \tag{3.4b}$$

$$-\partial\theta/\partial t + \nabla^2\theta + w = 0, \tag{3.4c}$$

where the dependent variables have been non-dimensionalized with $r_0$ (radius of the cylinder) for length, $r_0^2/\kappa$ for time, $\kappa/r_0$ for velocity, $\rho\nu\ \kappa/r_0^2$ for pressure and

$\gamma r_0$ for temperature. The equations (3.4) contain two dimensionless parameters: the Prandtl number defined as $Pr = \nu/\kappa$ and the Rayleigh number, defined as $R(t) = \alpha g(t)\, \gamma r_0^4/(\nu\kappa)$.

We consider the following boundary conditions on the sidewalls (at $r = 1$):

$$U = 0, \tag{3.5}$$

$$\partial\theta/\partial r = -b\theta, \tag{3.6a}$$

where $b$ is the Biot number. In the following we will mainly consider the two limit cases, corresponding to insulated or perfectly conducting sidewalls, respectively,

$$\partial\theta/\partial r = 0, \tag{3.6b}$$

or

$$\theta = 0. \tag{3.6c}$$

In addition, the solutions must be regular at $r = 0$.

## 4. Problem formulation

We consider disturbances of the form

$$U(r,\phi,z,t) = \mathrm{e}^{\mathrm{i}az+\mathrm{i}n\phi}\, U(r,t), \tag{4.1a}$$

$$\theta(r,\phi,z,t) = \mathrm{e}^{\mathrm{i}az+\mathrm{i}n\phi}\, \theta(r,t), \tag{4.1b}$$

and

$$p(r,\phi,z,t) = \mathrm{e}^{\mathrm{i}az+\mathrm{i}n\phi}\, p(r,t), \tag{4.1c}$$

where $a$ and $n$ denote axial and azimuthal wavenumbers. We also introduce two parameters, the mean Rayleigh number, $R_0$, and the vibrational Rayleigh number, $\epsilon$:

$$R_0 = \alpha g_0 \gamma r_0^4/(\nu\kappa), \tag{4.2a}$$

$$\epsilon = \alpha\epsilon^* \gamma r_0^4/(\nu\kappa) = \epsilon_{\mathrm{v}} R_0, \tag{4.2b}$$

where $\epsilon_{\mathrm{v}}$ is the usual non-dimensional modulation amplitude:

$$\epsilon_{\mathrm{v}} = \epsilon^*/g_0. \tag{4.2c}$$

The aim is to calculate the critical values of $R_0$ as a function of $\epsilon$ and $a$, for fixed $Pr$ and $\omega$. The minimum critical value of $R_0$, $R_0^{\mathrm{c}}$, is such that

$$\partial R_0(a,\epsilon)/\partial a = 0. \tag{4.3}$$

We denote by $a_{\mathrm{c}}$ the critical value of $a$ at which the condition (4.3) holds.

From a physical point of view, in the limit of small values of $\epsilon$, the effect of modulation is to alter the critical Rayleigh number, such that

$$R_0^{\mathrm{c}}(a_{\mathrm{c}},\epsilon) = R_{00}^{\mathrm{c}} + \eta_{\mathrm{c}}(a_{\mathrm{c}},\epsilon), \tag{4.4}$$

where $R_{00}^{\mathrm{c}}$ is the critical Rayleigh number in the unmodulated case ($\epsilon = 0$), and $\eta_{\mathrm{c}}$ is a function of $a_{\mathrm{c}}$ and $\epsilon$ such that

$$\eta_{\mathrm{c}}(a_{\mathrm{c}},0) = 0. \tag{4.5a}$$

Moreover, it can be shown that the change of $\epsilon$ into $-\epsilon$ corresponds to the translation of the time origin by a half period and, therefore, it does not change the physical problem. Thus, for small values of $\epsilon$, $\eta_{\mathrm{c}}$ is of the form

$$\eta_{\mathrm{c}}(a_{\mathrm{c}},\epsilon) = k\epsilon^2 + O(\epsilon^4). \tag{4.5b}$$

By using the Taylor expansion of $\partial R_0/\partial a$ and the power expansion of $a_c$:

$$a_c = a_0 + \epsilon a_1 + O(\epsilon^2),$$

Venezian (1969) demonstrated that $a_1 = 0$ and that $a_0$ is the critical wavenumber of the unmodulated case. In addition, for an infinite vertical cylinder. Yih (1959) proved that $a_0 = 0$. Then, we have

$$a_c = O(\epsilon^2). \tag{4.6}$$

The critical Rayleigh number (4.4) will thus be determined through (4.5*b*), in which the factor $k$ will be computed for the condition (4.6). However, this expression (4.5*b*) is not always the most convenient, as $\epsilon$ depends on $R_0^c$. We can give an explicit formulation of (4.5*b*) in terms of $\epsilon_v$; accounting for the definition (4.2*b*) of $\epsilon$, we have

$$\eta^2 + \eta[2R_{00}^c - (k\epsilon_v^2)^{-1}] + R_{00}^{c2} = 0.$$

This expression exhibits two solutions for any $\epsilon_v$ when $k < 0$. For $k > 0$ the solutions only exist in the domain $0 \leqslant \epsilon_v \leqslant \epsilon_v^0$, where $\epsilon_v^0 = 0.5\,(R_{00}^c k)^{-\frac{1}{2}}$. The two solutions are, for small $\epsilon_v$:

$$\eta_1 = kR_{00}^{c2}\,\epsilon_v^2, \tag{4.7a}$$

$$\eta_2 = -2R_{00}^c + (k\epsilon_v^2)^{-1} - kR_{00}^{c2}\,\epsilon_v^2. \tag{4.7b}$$

In practice, as $\eta_1 < \eta_2$, only the first family (4.7*a*) is interesting for the determination of the first instability threshold.

We restrict ourselves to the case of an infinitely long cylinder for which (after Gershuni & Zhukhovitskii 1963) $u, v$ and $p$ are $O(\epsilon)$. From the definition (4.1*c*), the term $\partial p/\partial z$ is proportional to $p$ and to $a_c$ (equal to $O(\epsilon^2)$ from (4.6)). This term is $O(\epsilon^3)$; it is cancelled, compared to the terms in $\epsilon^2$, in (3.4*a*). Thus, defining a vector field $\boldsymbol{X}$ with components $w(r,t)$ and $\theta(r,t)$, we obtain from (3.4),

$$\mathrm{D}\boldsymbol{X}/\mathrm{d}t = \boldsymbol{M}_0\boldsymbol{X} - \epsilon\cos\omega t\,\boldsymbol{N}\boldsymbol{X}, \tag{4.8}$$

with

$$\boldsymbol{M}_0 = \begin{pmatrix} Pr\,\mathrm{P}_n & -Pr\,R_0 \\ -1 & \mathrm{P}_n \end{pmatrix} \tag{4.9a}$$

and

$$\boldsymbol{N} = \begin{pmatrix} 0 & Pr \\ 0 & 0 \end{pmatrix}, \tag{4.9b}$$

where $\mathrm{P}_n$ is a linear operator defined by

$$\mathrm{P}_n = \partial^2/\partial r^2 + r^{-1}\partial/\partial r - n^2/r^2. \tag{4.10}$$

The use of Floquet Theory leads us to look for solutions of the following form:

$$\boldsymbol{X}(r,t) = \mathrm{e}^{\sigma t}\,\boldsymbol{x}(r,t), \tag{4.11}$$

such that $\boldsymbol{x}(r,t)$ is a periodic function in time. Here $\sigma$ is the so-called Floquet exponent. If we define the small parameter $\eta$ by

$$\eta = R_0 - R_{00}, \tag{4.12}$$

where the double index 00 represents the unmodulated conditions, substitution of (4.11) and (4.12) into (4.8) gives the equation

$$(\boldsymbol{M}_{00} - \mathrm{d}/\mathrm{d}t)\,\boldsymbol{x} = \sigma\boldsymbol{x} + \eta\boldsymbol{N}\boldsymbol{x} + \epsilon\cos\omega t\,\boldsymbol{N}\boldsymbol{x}, \tag{4.13}$$

where $\boldsymbol{M}_{00}$ is defined by

$$\boldsymbol{M}_{00} = \begin{pmatrix} Pr\,\mathrm{P}_n & -Pr\,R_{00} \\ -1 & \mathrm{P}_n \end{pmatrix}. \tag{4.14}$$

In order to determine the critical stability conditions, we suppose that the effect of the modulation is weak when the amplitude $\epsilon$ is small. Thus, the parameter $\eta$ remains small and the solutions at $O(\epsilon, \eta)$ are those of the unmodulated case,

$$\boldsymbol{x} = \boldsymbol{x}_{00} + O(\epsilon, \eta), \tag{4.15a}$$

$$\sigma = \sigma_{00} + O(\epsilon, \eta), \tag{4.15b}$$

where $\boldsymbol{x}_{00}$ is the solution of the eigenvalue problem

$$(\boldsymbol{M}_{00} - \mathrm{d}/\mathrm{d}t)\,\boldsymbol{x}_{00} = \sigma_{00}\,\boldsymbol{x}_{00}, \tag{4.16}$$

with the appropriate boundary conditions.

According to the principle of exchange of stability, $\sigma_{00} = 0$ is a simple eigenvalue of (4.16). In order to investigate the effect of $\epsilon$ and $\eta$ on the solution of (4.13) we assume

$$\boldsymbol{x} = \boldsymbol{x}_{00} + \sum_{p+q \geqslant 1} (\epsilon^p \eta^q \boldsymbol{x}_{pq}), \tag{4.17a}$$

$$\sigma = \sum_{p+q \geqslant 1} (\epsilon^p \eta^q \sigma_{pq}). \tag{4.17b}$$

Here the critical conditions are obtained for a certain function $\eta = \eta_c(\epsilon)$ such that the real part of $\sigma$ is zero,

$$\mathrm{Re}\,\{\sigma(\epsilon, \eta_c(\epsilon))\} = 0. \tag{4.18}$$

Substitution of (4.17) into (4.13) and identification of coefficients of like powers of $\epsilon, \eta$ and $\epsilon^2$ leads to the following recursive equations:

$$(\boldsymbol{M}_{00} - \mathrm{d}/\mathrm{d}t)\,\boldsymbol{x}_{00} = 0, \tag{4.19a}$$

$$(\boldsymbol{M}_{00} - \mathrm{d}/\mathrm{d}t)\,\boldsymbol{x}_{10} = \sigma_{10}\,\boldsymbol{x}_{00} + \cos\omega t\,\boldsymbol{N}\boldsymbol{x}_{00}, \tag{4.19b}$$

$$(\boldsymbol{M}_{00} - \mathrm{d}/\mathrm{d}t)\,\boldsymbol{x}_{01} = \sigma_{01}\,\boldsymbol{x}_{00} + \boldsymbol{N}\boldsymbol{x}_{00}, \tag{4.19c}$$

$$(\boldsymbol{M}_{00} - \mathrm{d}/\mathrm{d}t)\,\boldsymbol{x}_{20} = \sigma_{20}\,\boldsymbol{x}_{00} + \sigma_{10}\,\boldsymbol{x}_{10} + \cos\omega t\,\boldsymbol{N}\boldsymbol{x}_{10}. \tag{4.19d}$$

The first equation corresponds to the unmodulated case. In order for the other equations to have periodic solutions, the steady part of the right-hand side must be orthogonal to the null space of the adjoint operator of $\boldsymbol{M}_{00}$. Denoting by $\langle \boldsymbol{X} \mid \boldsymbol{Y} \rangle$ the scalar product, the solvability conditions of the equations (4.19) are

$$\sigma_{10} = 0,$$

$$\sigma_{01} = -\langle \boldsymbol{N}\boldsymbol{x}_{00} \mid \boldsymbol{x}^*_{00} \rangle / \langle \boldsymbol{x}_{00} \mid \boldsymbol{x}^*_{00} \rangle, \tag{4.20a}$$

$$\sigma_{20} = -\langle \overline{\cos\omega t\,\boldsymbol{N}\boldsymbol{x}_{10} \mid \boldsymbol{x}^*_{00}} \rangle / \langle \boldsymbol{x}_{00} \mid \boldsymbol{x}^*_{00} \rangle, \tag{4.20b}$$

where the bar denotes a time average. The expansion (4.17*b*) reads

$$\sigma = \eta\sigma_{01} + \epsilon^2 \sigma_{20},$$

and the condition (4.18) becomes, as $\sigma_{01}$ is real,

$$\eta_c = -\mathrm{Re}(\sigma_{20})/\sigma_{01}\,\epsilon^2 + O(\epsilon^4).$$

Therefore, according to the notation (4.5$b$), we have

$$k = -\mathrm{Re}(\sigma_{20})/\sigma_{01}. \tag{4.21}$$

Representing $\boldsymbol{x}_{10}$ in the form

$$\boldsymbol{x}_{10} = \boldsymbol{x}_{10}^{1}\,\mathrm{e}^{\mathrm{i}\omega t} + \langle\text{conjugate}\rangle, \tag{4.22}$$

we have, from (4.19$b$), to solve the following problem for $\boldsymbol{x}_{10}^{1}$:

$$(\boldsymbol{M}_{00} - \mathrm{i}\omega)\,\boldsymbol{x}_{10}^{1} = 0.5\,\boldsymbol{N}\boldsymbol{x}_{00}. \tag{4.23}$$

The system (4.19) will be solved in the following paragraphs with two different techniques based on the use of Bessel functions. An analytical approach is considered in the following paragraph for the adiabatic boundary conditions, while a more general method, called matrix technique, is presented in §6.

## 5. Analytical approach (adiabatic case)

In this section we consider an analytical approach to solve the problems (4.19). In order to alleviate the presentation of the method we will consider the adiabatic conditions (3.6$b$) only. For the unmodulated case (4.19$a$), we have

$$w_{00} = I_n(\xi r) - \mu\, J_n(\xi r), \tag{5.1}$$

$$\theta_{00} = [I_n(\xi r) + \mu\, J_n(\xi r)]\,\xi^{-2}, \tag{5.2}$$

where

$$\xi = R_{00}^{\frac{1}{4}}, \quad \mu = I_n(\xi)/J_n(\xi). \tag{5.3}$$

$J_n$ and $I_n$ are respectively, the Bessel function and the modified Bessel function of order $n$. The critical values of $\xi$ are the ones satisfying the following characteristic relation:

$$I_n(\xi)\,J_n'(\xi) + I_n'(\xi)\,J_n(\xi) = 0. \tag{5.4}$$

where the prime denotes a first derivative. Then, the critical value of the Rayleigh number for the unmodulated case is given by

$$R_{00}^{\mathrm{c}} = (\xi^{\mathrm{c}})^4. \tag{5.5}$$

For $n = 0$ and $n = 1$ respectively, we have the classical solutions

$$\xi^{\mathrm{c}} = 4.611, \quad R_{00}^{\mathrm{c}} = 452.1, \tag{5.6a}$$

$$\xi^{\mathrm{c}} = 2.871, \quad R_{00}^{\mathrm{c}} = 67.9. \tag{5.6b}$$

### 5.1. *General case (finite $\omega$)*

In order to solve the system (4.19) we introduce the scalar product

$$\langle \boldsymbol{X} | \boldsymbol{Y} \rangle = \int_0^1 \boldsymbol{X}\cdot\boldsymbol{Y}\,\mathrm{d}r.$$

In fact, it is convenient to multiply all the equations (4.19) by $r$, since $r\mathrm{P}_n$ is self-adjoint. The adjoint operator of $r\boldsymbol{M}_{00}$ is

$$(r\boldsymbol{M}_{00})^{*} = \begin{pmatrix} rPr\,\mathrm{P}_n & -r \\ -rPr\,R_{00} & r\mathrm{P}_n \end{pmatrix}.$$

The null space of $(r\boldsymbol{M}_{00})^*$ is generated by the vector $\boldsymbol{x}^*_{00}$ of components

$$w^*_{00} = w_{00}, \quad \theta^*_{00} = Pr\,\mathrm{P}_n\, w_{00}.$$

The solvability conditions (4.20) of the equations (4.19), multiplied by $r$, are

$$\sigma_{01} = \frac{\displaystyle\int r\boldsymbol{N}\boldsymbol{x}_{00}\cdot\boldsymbol{x}^*_{00}\,\mathrm{d}r}{\displaystyle\int r\boldsymbol{x}_{00}\cdot\boldsymbol{x}^*_{00}\,\mathrm{d}r}, \tag{5.7a}$$

$$\sigma_{20} = \frac{\displaystyle\int r\overline{(\cos\omega t\,\boldsymbol{N}\boldsymbol{x}_{10})}\cdot\boldsymbol{x}^*_{00}\,\mathrm{d}r}{\displaystyle\int r\boldsymbol{x}_{00}\cdot\boldsymbol{x}^*_{00}\,\mathrm{d}r}. \tag{5.7b}$$

To $O(\epsilon^4, \eta^2)$, owing to the form of $N$, we only need to know the $\theta^1_{10}$ component of $\boldsymbol{x}^1_{10}$, which has to satisfy the equation (4.23). We have to solve the following problem for $\theta^1_{10}$:

$$(\mathrm{P}^2_n - \lambda\mathrm{P}_n - \beta)\,\theta^1_{10} = \tfrac{1}{2}\theta_{00}, \tag{5.8}$$

with the two boundary conditions

$$\partial\theta^1_{10}/\partial r = 0, \quad \mathrm{P}_n\,\theta^1_{10} = 0 \quad \text{at}\, r = 1, \tag{5.9}$$

where $\lambda$ and $\beta$ are defined by

$$\lambda = \mathrm{i}\omega(1+Pr^{-1}), \quad \beta = \omega^2 Pr^{-1} + \xi^4.$$

Note that here and in the following, $\xi$ will represent $\xi^c$.
If the discriminant of (5.8) is different from 0, i.e. ,

$$\lambda^2 + 4\beta \neq 0 \quad \text{or}\, \omega^2(1-Pr^{-1})^2 - 4\xi^4 \neq 0, \tag{5.10}$$

the general solution of (5.8) is

$$\theta^1_{10} = \alpha_1 I_n(\gamma_1 r) + \alpha_2 J_n(\gamma_2 r) + \beta_1 I_n(\xi r) + \beta_2 J_n(\xi r), \tag{5.11}$$

where $\beta_1$, $\beta_2$, $\gamma_1$ and $\gamma_2$ are defined by

$$\left.\begin{aligned} \beta_1 &= -[2\xi^2(\omega^2 Pr^{-1} + \lambda\xi^2)]^{-1}, \\ \beta_2 &= \mu[2\xi^2(-\omega^2 Pr^{-1} + \lambda\xi^2)]^{-1}, \\ \gamma_1 &= \{\tfrac{1}{2}[\lambda + (\lambda^2+4\beta)^{\frac{1}{2}}]\}^{\frac{1}{2}}, \\ \gamma_2 &= \{\tfrac{1}{2}[-\lambda + (\lambda^2+4\beta)^{\frac{1}{2}}]\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{5.12}$$

Then, the boundary conditions (5.9) give

$$\alpha_1 = (N1 - N2)/D1, \tag{5.13}$$

and
$$\alpha_2 = -[\xi\beta_1 I'_n(\xi) + \xi\beta_2 J'_n(\xi) + \gamma_1 I'_n(\gamma_1)\,\alpha_1]\,[\gamma_2 J'_n(\gamma_2)]^{-1} \tag{5.14}$$

where
$$\begin{aligned} N1 &= (\mathrm{i}\omega - \xi^2)\,\beta_1 I_n(\xi) + (\mathrm{i}\omega + \xi^2)\,\beta_2 J_n(\xi), \\ N2 &= (\gamma^2_2 + \mathrm{i}\omega)\,J_n(\gamma_2)\,\xi[\beta_1 I'_n(\xi) + \beta_2 J'_n(\xi)]\,[\gamma_2 J'_n(\gamma_2)]^{-1}, \\ D1 &= (\gamma^2_1 - \mathrm{i}\omega)\,I_n(\gamma_1) + (\gamma^2_2 + \mathrm{i}\omega)\,J_n(\gamma_2)\,[\gamma_1 I'_n(\gamma_1)]\,[\gamma_2 J'_n(\gamma_2)]^{-1}. \end{aligned}$$

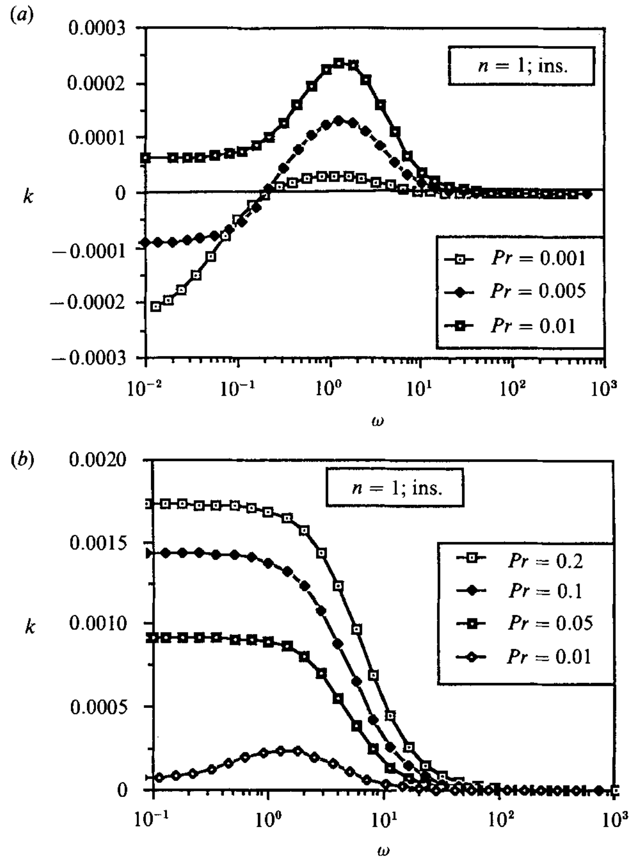


FIGURE 1 ($a$, $b$). For caption see facing page.

Finally, the expression of $k$ from (4.21) is

$$k=\frac{-\xi^2\,\mathrm{Re}\{\alpha_1(K_1-\mu K_1')+\alpha_2(K_2-\mu K_2')+\beta_1(K_3-\mu K_4)+\beta_2(K_4-\mu K_4')\}}{(K_3-\mu^2K_4')}, \tag{5.15}$$

where

$$\left.\begin{aligned}
K_1&=\int rI_n(\gamma_1 r)\,I_n(\xi r)\,\mathrm{d}r, & K_1'&=\int rI_n(\gamma_1 r)\,J_n(\xi r)\,\mathrm{d}r,\\
K_2&=\int rJ_n(\gamma_2 r)\,I_n(\xi r)\,\mathrm{d}r, & K_2'&=\int rJ_n(\gamma_2 r)\,J_n(\xi r)\,\mathrm{d}r,\\
K_3&=\int rI_n^2(\xi r)\,\mathrm{d}r, & K_4&=\int rI_n(\xi r)\,J_n(\xi r)\,\mathrm{d}r,\\
K_4'&=\int rJ_n^2(\xi r)\,\mathrm{d}r
\end{aligned}\right\} \tag{5.16}$$

These integrals (5.16) have been solved analytically. The computation of (5.15) requires an accurate evaluation of $\xi$ when solving (5.4) and an accurate determination of the Bessel functions occurring in (5.13)–(5.14) and (5.16), mainly when $\omega \rightarrow 0$. The method has been presented for insulated walls (3.6$b$), but it could be extended to the case of perfectly conducting walls (3.6$c$). The values of $k$ have been calculated from

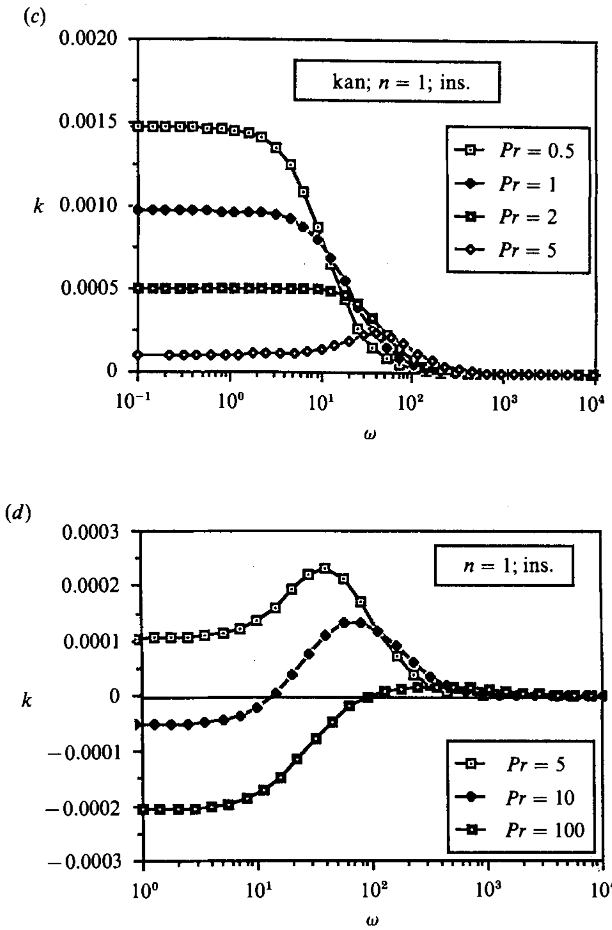


FIGURE 1. Alteration factor *vs.* frequency, for the insulated case at (*a*) $10^{-3} \leqslant Pr \leqslant 10^{-2}$; (*b*) $10^{-2} \leqslant Pr \leqslant 2\times10^{-1}$; (*c*) $0.5 \leqslant Pr \leqslant 5$; (*d*) $5 \leqslant Pr \leqslant 100$.

(5.15) in the case $n = 1$ for a wide range of $\omega$ values and for $10^{-3} \leqslant Pr \leqslant 10^{3}$. The results are presented in figures 1(*a*)–1(*d*), respectively for $10^{-3} \leqslant Pr \leqslant 10^{-2}$, $10^{-2} \leqslant Pr \leqslant 2.10^{-1}$, $5.10^{-1} \leqslant Pr \leqslant 5$ and $5 \leqslant Pr \leqslant 100$. These figures show that for the lowest and the highest $Pr$ values, $k$ becomes negative for small $\omega$, indicating that in this case the gravity modulation diminishes the critical value of the Rayleigh number for the onset of the convection compared to the unmodulated case, i.e. $R_0^c < R_{00}^c$. As $k$ appears to tend asymptotically to a minimum when $\omega \rightarrow 0$, we considered its limit value $k(0)$, to specify the domains of $Pr$ for which $k$ can take negative values (figure 2). In these domains (which correspond to $Pr < 8\times10^{-3}$ and $Pr > 8$) $k$ is negative from $\omega = 0$ up to a limit value of $\omega$ which depends on $Pr$; typically this limit is $\omega \sim 0.1$ for $Pr < 8\times10^{-3}$ (figure 1*a*) and $\omega \sim Pr$ for $Pr > 8$ (figure 1*d*).

For high $\omega$, $k$ is positive for any $Pr$ and goes to zero when $\omega \rightarrow \infty$. This behaviour is better illustrated by the log–log plotting in figures 3(*a*) and 3(*b*), respectively for $0.01 \leqslant Pr \leqslant 0.2$ and $0.5 \leqslant Pr \leqslant 5$, which shows that the value of $k$ behaves as $\omega^{-2}$ when $\omega \rightarrow \infty$. This property will be demonstrated in §5.2, for any $Pr$, in the specific case of the insulated wall and, under more general hypotheses, in §7.

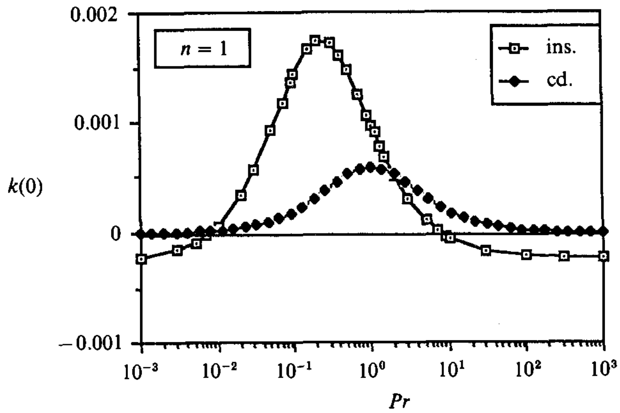


FIGURE 2. Alteration factor at $\omega = 0$ *vs*. Prandtl number, for insulated and conducting cases.

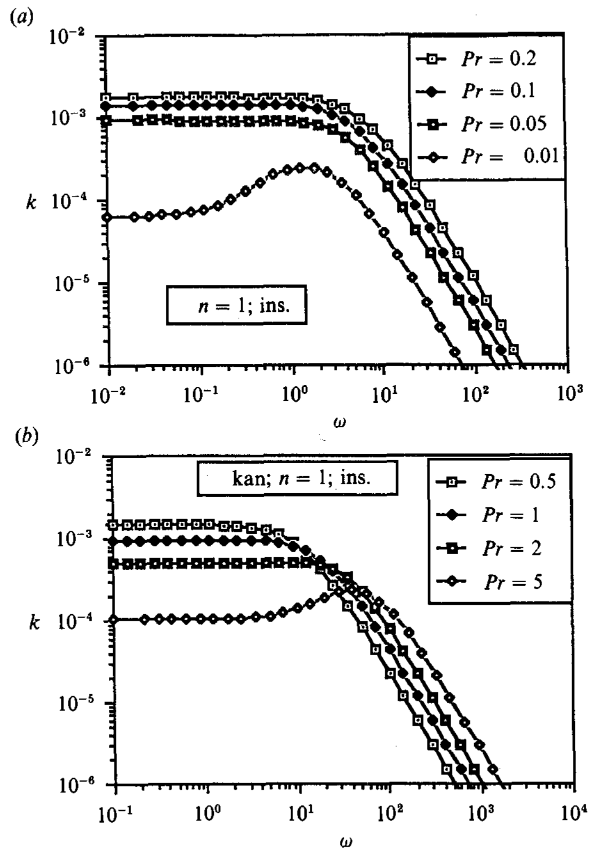


FIGURE 3. Alteration factor *vs*. frequency, for the insulated case at (*a*) $0.01 \leqslant Pr \leqslant 0.2$; (*b*) $0.5 \leqslant Pr \leqslant 5$.

Finally, we can remark that the condition (5.10) is always satisfied for $Pr = 1$. But for $Pr \neq 1$, this condition is not satisfied for

$$\omega = \omega_{\mathrm{d}} = 2\xi Pr/|Pr-1|. \tag{5.17}$$

For such a frequency the two first terms of the right-hand side of (5.11) are no longer linearly independent. A specific study would have to be done in that case, but in fact we observe that the expression (5.15) gives continuous results even for $\omega = \omega_d$. This continuity means that the value $\omega = \omega_d$ is not a singular point; this feature will be confirmed by the results of the matrix method developed in §6.

### 5.2. *Limiting case*: $\omega \to \infty$

If we use the asymptotic representation of Bessel functions for large values of the argument, i.e.

$$I_n(x) = e^x[1+O(x^{-1})]\,(2\pi x)^{-\frac{1}{2}}, \tag{5.18}$$

and

$$J_n(x) = [\cos(x-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi)+O(x^{-1})]\,(\tfrac{1}{2}x\pi)^{-\frac{1}{2}}, \tag{5.19}$$

we find that the most important term in the numerator of $k$ in (5.15) is

$$\mathrm{Re}\{\beta_1 K_3-\mu\beta_2 K_4'\}.$$

Thus,

$$k = \omega^2/[2Pr(\omega^4 Pr^{-2}-\lambda^2\xi^2)]+O(\omega^{-4}) \tag{5.20a}$$

$$= 0.5\,Pr/\omega^2+O(\omega^{-4}). \tag{5.20b}$$

This relation, (5.20*b*), confirms the behaviour seen in figures 3(*a*) and 3(*b*) and in addition, shows that $k$ is proportional to $Pr$ for large $\omega$. Thus, the critical Rayleigh number for large $\omega$ is

$$R_0^c = R_{00}^c+0.5\,Pr[\epsilon/\omega]^2. \tag{5.21}$$

It is always greater than $R_{00}^c$ which is given by (5.5).

It can be noted that (5.20) and (5.21) have been established under the condition (5.10), which is discussed at the end of §5.1. For the large values of $\omega$ considered in this paragraph, the condition (5.10) is always satisfied. We will recover the relation (5.21) in a quite general case, by a direct asymptotic analysis presented in §7.

## 6. Matrix method

Our aim in this section is to present a general method for solving the system (4.8) by the use of trial functions. The advantage of this method is that it can be easily extended to general boundary conditions and geometries, especially for confined cylinders (finite length). Here, the case of infinite cylinders only is considered, with either conducting or insulated walls. We mainly focus our attention to the case $n = 1$, which corresponds to the basic mode of instability for long cylinders (Ostroumov 1952; Yih 1959; Charlson & Sani 1971). Then the system (4.8) can be written

$$Pr^{-1}\frac{\partial w}{\partial t} = \mathrm{P}_1 w-R\theta, \tag{6.1}$$

$$\frac{\partial \theta}{\partial t} = -w+\mathrm{P}_1\theta, \tag{6.2}$$

with the boundary conditions (3.5) and (3.6).

The linear operator $\mathrm{P}_1$ in the equations (6.1) and (6.2) is defined from (4.10) as

$$\mathrm{P}_1 = \partial^2/\partial r^2+r^{-1}\partial/\partial r-r^{-2}. \tag{6.3}$$

We use the expansions

$$w = \sum_i a_i(t)\,w_i(r), \tag{6.4}$$

$$\theta = \sum_i b_i(t)\,\theta_i(r), \tag{6.5}$$

where the trial functions $w_i$ and $\theta_i$ satisfy the equations

$$\mathrm{P}_1 w_i = -\xi_i^2 w_i, \tag{6.6}$$

$$\mathrm{P}_1 \theta_i = -\gamma_i^2 \theta_i, \tag{6.7}$$

and the boundary conditions (3.5) and (3.6). The solutions are

$$w_i(r) = J_1(\xi_i r), \tag{6.8}$$

$$\theta_i(r) = J_1(\gamma_i r), \tag{6.9}$$

$J_1$ being the first-order Bessel function, and $\xi_i$ and $\gamma_i$ satisfying the relations

$$J_1(\xi_i) = 0, \tag{6.10}$$

and $$J_1'(\gamma_i) = 0, \quad \text{for insulated walls,} \tag{6.11a}$$

or $$J_1(\gamma_i) = 0 \quad \text{for conducting walls,} \tag{6.11b}$$

or $$\gamma_i J_1'(\gamma_i) + b J_1(\gamma_i) = 0 \quad \text{for finite Biot number.} \tag{6.11c}$$

By substituting (6.4) and (6.5) into (6.1) and (6.2) respectively, and multiplying these equations by $rw_j$ and $r\theta_j$, and then integrating on the interval $r \in [0, 1]$, we obtain the system

$$\boldsymbol{K}\,\mathrm{d}\boldsymbol{X}/\mathrm{d}t = \boldsymbol{M}'\boldsymbol{X}, \tag{6.12}$$

with $$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_1 & 0 \\ 0 & \boldsymbol{K}_2 \end{pmatrix}, \quad \boldsymbol{M}' = \begin{pmatrix} \boldsymbol{A}_1 & -R\boldsymbol{C} \\ -{}^{\mathrm{t}}\boldsymbol{C} & \boldsymbol{A}_2 \end{pmatrix}, \tag{6.13}$$

where matrices $\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{A}_1, \boldsymbol{A}_2$, and $\boldsymbol{C}$ are defined by their general term

$$\begin{cases} K_{1ij} = -\displaystyle\int_0^1 r w_i w_j \,\mathrm{d}r, \quad K_{2ij} = \int_0^1 r\theta_i \theta_j \,\mathrm{d}r, \quad C_{ij} = \int_0^1 r\theta_i w_j \,\mathrm{d}r, \\ A_{1ij} = \displaystyle\int_0^1 r\mathrm{P}_n w_i w_j \,\mathrm{d}r, \quad A_{2ij} = \int_0^1 r\mathrm{P}_n \theta_i \theta_j \,\mathrm{d}r, \end{cases}$$

and where ${}^{\mathrm{t}}\boldsymbol{C}$ denotes the matrix transpose of $\boldsymbol{C}$.

### 6.1. *General thermal boundary conditions*

As in §4, we use the parameter $\eta$ and the Floquet exponent $\sigma$. Thus the perturbation equations (6.12) can be written as

$$(\boldsymbol{M}'_{00} - \mathrm{d}/\mathrm{d}t)\,\boldsymbol{x} = \sigma\boldsymbol{x} + \eta\boldsymbol{N}'\boldsymbol{x} + \epsilon \cos\omega t\,\boldsymbol{N}'\boldsymbol{x}, \tag{6.14}$$

where $\boldsymbol{M}'_{00}$ and $\boldsymbol{N}'$ are defined by

$$\boldsymbol{M}'_{00} = \begin{pmatrix} \boldsymbol{K}_1^{-1}\boldsymbol{A}_1 & -R_{00}\boldsymbol{K}_1^{-1}\boldsymbol{C} \\ -\boldsymbol{K}_2^{-1}\,{}^{\mathrm{t}}\boldsymbol{C} & \boldsymbol{K}_2^{-1}\boldsymbol{A}_2 \end{pmatrix}, \quad \boldsymbol{N}' = \begin{pmatrix} 0 & \boldsymbol{K}_1^{-1}\boldsymbol{C} \\ 0 & 0 \end{pmatrix}, \tag{6.15}$$

where $\boldsymbol{K}_1^{-1}$ and $\boldsymbol{K}_2^{-1}$ represent the inverses of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$.

We also look for solutions of the form (4.17) and get recursive equations of the form (4.19). The solvability condition (4.20) for these equations reads

$$\sigma_{01} = \langle \boldsymbol{x}_{00}^* \,|\, \boldsymbol{N}'\boldsymbol{x}_{00} \rangle / \langle \boldsymbol{x}_{00}^* \,|\, \boldsymbol{x}_{00} \rangle, \tag{6.16a}$$

$$\sigma_{20} = 0.5\,\langle \boldsymbol{x}_{00}^* \,|\, \{\boldsymbol{N}'(\boldsymbol{M}'_{00} + \mathrm{i}\omega\boldsymbol{I})^{-1}\}\,\boldsymbol{N}'\boldsymbol{x}_{00} \rangle / \langle \boldsymbol{x}_{00}^* \,|\, \boldsymbol{x}_{00} \rangle. \tag{6.16b}$$

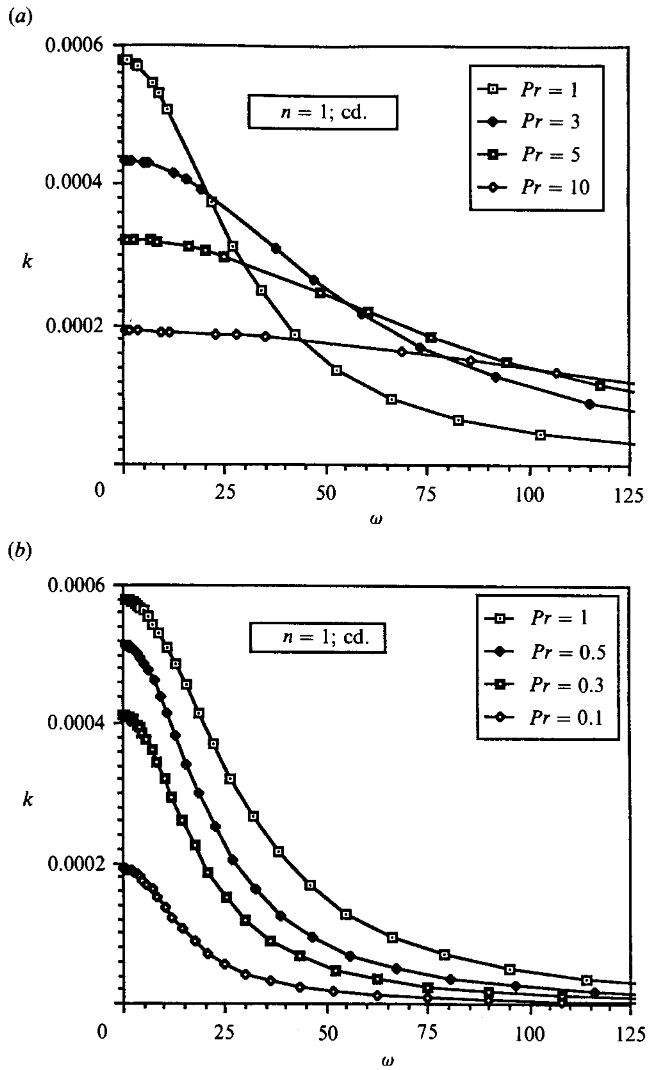


FIGURE 4. Alteration factor *vs.* frequency, for the conducting case at (*a*) $1 \leqslant Pr \leqslant 10$; (*b*) $0.1 \leqslant Pr \leqslant 1$.

In these expressions the symbol $\langle \, | \, \rangle$ denotes the standard scalar product, $\boldsymbol{I}$ is the identity matrix,

$$\boldsymbol{x}_{00} \text{ is a solution of } \boldsymbol{M}'_{00}\,\boldsymbol{x}_{00} = 0,$$

and

$$\boldsymbol{x}^*_{00} \text{ is a solution of } {}^{t}\boldsymbol{M}'_{00}\,\boldsymbol{x}^*_{00} = 0.$$

Thus, from the expressions (6.16*a*, *b*), we can compute the values of $k$ defined by (4.21). These computations depend on $R^{c}_{00}$ (the classical critical Rayleigh number for the unmodulated case), values of which are, respectively, 67.963 and 215.560 for the insulated and conducting cases. The computations which have been done for a wide range of $\omega$ and $Pr$ values need a highly accurate algorithm to invert matrices, mainly for insulated walls. The influence of the number of trial functions $L$ has been checked; most of the computations have been done with $L = 30$ for insulated walls, while $L = 5$ appeared sufficient for conducting walls.

For insulated walls, the results recover well those of the analytical method of §4; they would give exactly the same graphs as in figures 1–3 and thus are not repeated here.

For the conducting case, the computations carried out in the range $0 < \omega < 10^4$ are plotted in figures 4(*a*) and 4(*b*) respectively, for $0.1 < Pr < 1$ and $1 < Pr < 10$; they show that $k$ is always positive and reaches a maximum when $\omega \to 0$. The limit value of $k$ when $\omega \to 0$, denoted $k(0)$, is also presented in figure 2; this graph, which is plotted as a function of $Pr$, shows a 'symmetry' with respect to $Pr = 1$ (the results being identical for $Pr$ and $1/Pr$) when $\omega \to 0$.

### 6.2. *Particular case ($n = 1$, conducting walls)*

Due to the symmetry existing in the case $n = 1$, for conducting walls, the matrices $\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{A}_1, \boldsymbol{A}_2$ and $\boldsymbol{C}$ are diagonal and thus the expressions (6.15) can be considerably simplified. The expressions (6.16) can be analytically determined after some algebra and, finally, $k$ emerges as

$$k = 0.5\, Pr[\omega^2 + R^{\mathrm{c}}_{00}((Pr+1)^2]^{-1}. \tag{6.17}$$

For a given $Pr$, $k$ reaches a maximum when $\omega \to 0$. The limiting form of (6.17) is then

$$k(0) = 0.5\, R^{\mathrm{c}-1}_{00}\,(Pr^{-\frac{1}{2}} + Pr^{\frac{1}{2}})^{-2}. \tag{6.18}$$

As already mentioned, it exhibits 'symmetry' with respect to $Pr = 1$ (see figure 2). The relation (6.17) has been used to control the validity of the numerical code used to calculate the expressions of (6.16) and $k$ through (4.21), for the entire domain $0 < \omega < \infty$. The agreement is very good, the three first decimal places being identical. This confirms the results plotted in figure 4(*a*, *b*) and proves that $k$ is positive for any $Pr$ and $\omega$. This contrasts with the adiabatic case, where $k$ takes negative values for small $\omega$, when $Pr$ is small or large enough (see figure 2 and figures 1(*a*)–(*d*)).

When $\omega$ tends to infinity, the expression (6.17) for conducting walls recovers the relation (5.20*b*) established for the insulated case, indicating no effect of the thermal boundary conditions in the alteration factor $k$, under gravity modulation. This property will be confirmed for more general conditions in the following paragraph.

## 7. Asymptotic analysis for large values of $\omega$

We present in this section a general asymptotic analysis of equations of the following form:

$$\mathrm{d}X/\mathrm{d}t = \boldsymbol{M}_0 X + \epsilon \cos \omega t \boldsymbol{N} X, \tag{7.1}$$

in which $\boldsymbol{M}_0$ has distinct eigenvalues and $\boldsymbol{N}$ is a nilpotent matrix (i.e. $\boldsymbol{N}^2 = 0$). Of course, these conditions for $\boldsymbol{M}_0$ and $\boldsymbol{N}$ were already satisfied in the equation (4.18) used in the previous sections. But they apply for rather more general situations (other kinds of periodic modulations or geometries); they are satisfied for most of the periodic modulations usually considered in the literature (gravity modulation or temperature modulation of surfaces). Note, in addition, that equation (7.1) is of the same type as (6.12) if $\boldsymbol{K}$ is invertible.

The present analysis is developed for large $\omega$. However, in contrast with the previous paragraphs where $\epsilon$ was assumed to be small, we now consider that $\epsilon$ may increase, but such that

$$\epsilon/\omega \ll 1. \tag{7.2}$$

Venezian (1969) also developed an asymptotic analysis; he, however, assumed that $\epsilon$ remains small and explains the reasons why the inequality (7.2) limits the validity of his own results. Another asymptotic analysis has been proposed by Gershuni *et al.* (1970) with finite amplitude modulations but in the framework of a separation of variables solution; they derived an analytical relation that will be discussed further.

We now introduce a new small parameter

$$\delta = \epsilon/\omega, \tag{7.3}$$

and we look for solutions of the form

$$\boldsymbol{X} = \exp\left[u_1(t)\,\boldsymbol{N} + u_2(t)\,\boldsymbol{N}_2\right]\boldsymbol{Y}, \tag{7.4}$$

with $$u_1(t) = \delta \sin \omega t, \quad u_2(t) = \delta^2 \sin 2\omega t/4\omega, \quad \boldsymbol{N}_2 = \boldsymbol{N}\boldsymbol{M}_0\boldsymbol{N}. \tag{7.5}$$

Then, the equation (7.1) can be written:

$$\mathrm{d}\boldsymbol{Y}/\mathrm{d}t = \boldsymbol{L}_0\,\boldsymbol{Y} - (u_1\,\boldsymbol{M}_2 - u_2\,\boldsymbol{M}_3)\,\boldsymbol{Y} - (2u_1\,u_2\,\boldsymbol{N}_2\,\boldsymbol{M}_0\,\boldsymbol{N} + u_2^2\,\boldsymbol{N}_2\,\boldsymbol{M}_0\,\boldsymbol{N}_2)\,\boldsymbol{Y}, \tag{7.6}$$

with $\boldsymbol{L}_0 = \boldsymbol{M}_0 - 0.5\delta^2\boldsymbol{N}_2,\quad \boldsymbol{M}_2 = \boldsymbol{N}\boldsymbol{M}_0 - \boldsymbol{M}_0\boldsymbol{N},\quad \boldsymbol{M}_3 = \boldsymbol{N}_2\,\boldsymbol{M}_0 - \boldsymbol{M}_0\,\boldsymbol{N}_2.$

At $O(\delta^3)$ equation (7.6) reads

$$\mathrm{d}\boldsymbol{Y}/\mathrm{d}t = \boldsymbol{L}_0\,\boldsymbol{Y} - \delta \sin \omega t\,\boldsymbol{M}_2\,\boldsymbol{Y} + \delta^2 \sin 2\omega t/4\omega\,\boldsymbol{M}_3\,\boldsymbol{Y}. \tag{7.7}$$

This equation (7.7) can be solved as in the previous sections by using $\eta$ and $\delta$ instead of $\epsilon$:

$$\boldsymbol{L}_0 = \boldsymbol{L}_{00} - \eta\boldsymbol{N},$$

$$\boldsymbol{Y} = \boldsymbol{Y}_{00} + \sum_{p+q\geqslant 1} \delta^p\eta^q\,\boldsymbol{Y}_{pq}, \tag{7.8}$$

$$\sigma = \sigma_{00} + \sum_{p+q\geqslant 1} \delta^p\eta^q\,\sigma_{pq}, \tag{7.9}$$

where $\boldsymbol{L}_{00}$ is obtained from $\boldsymbol{L}_0$ by taking $R_{00}$ instead of $R_0$.

When the principle of exchange of stability is valid, critical conditions are obtained for

$$\eta_c = -\{0.5\,\langle \boldsymbol{N}_2\boldsymbol{Y}_{00} \mid \boldsymbol{Y}^*_{00}\rangle + \langle\overline{\sin\omega t\,\boldsymbol{M}_2\,\boldsymbol{Y}_{10} \mid \boldsymbol{Y}^*_{00}}\rangle\}\,\delta^2/\langle \boldsymbol{N}\boldsymbol{Y}_{00} \mid \boldsymbol{Y}^*_{00}\rangle + O(\delta^4), \tag{7.10}$$

where the bar represents time averaging.

If we remark that the second term in (7.10) is $O(\omega^{-2})$, since

$$\overline{\sin\omega t\,\boldsymbol{M}_2\,\boldsymbol{Y}_{10}} = 0.5\,\boldsymbol{M}_2\,\mathrm{Re}\,(\boldsymbol{L}_{00} - \mathrm{i}\omega)^{-1}\,\boldsymbol{M}_2\,\boldsymbol{Y}_{00} \tag{7.11a}$$

$$= 0.5\,\boldsymbol{M}_2\,\boldsymbol{L}_{00}[(\boldsymbol{L}_{00})^2 + \omega^2]^{-1}\,\boldsymbol{M}_2\,\boldsymbol{Y}_{00}, \tag{7.11b}$$

we have

$$\eta_c = -\{0.5\,\langle \boldsymbol{N}_2\,\boldsymbol{Y}_{00} \mid \boldsymbol{Y}^*_{00}\rangle/\langle \boldsymbol{N}\boldsymbol{Y}_{00} \mid \boldsymbol{Y}^*_{00}\rangle + O(\omega^{-2})\}\,\delta^2 + O(\delta^4). \tag{7.12}$$

This is quite a 'universal' result in the sense that it applies for any kind of fluid layer geometries and periodic constraints, under the assumptions only that $\boldsymbol{N}^2 = 0$ and that the principle of exchange of stability is valid. With appropriate operators, this analysis can also be applied to bounded domains.

In the case of an infinite vertical cylinder, as considered in the previous sections, we have from (7.5)

$$\boldsymbol{N}_2 = \begin{pmatrix} 0 & Pr \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Pr\,\mathrm{P}_n & -Pr\,R_0 \\ -1 & \mathrm{P}_n \end{pmatrix}\begin{pmatrix} 0 & Pr \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -Pr^2 \\ 0 & 0 \end{pmatrix} = -Pr\,\boldsymbol{N}. \tag{7.13}$$

Substitution of (7.13) into (7.12) gives

$$\eta_c = [0.5\,Pr + O(\omega^{-2})]\,\delta^2 + O(\delta^4),$$

or, in terms of $\epsilon$

$$\eta_c = \epsilon^2[0.5\,Pr\,\omega^{-2} + O(\omega^{-4})]. \tag{7.14}$$

Finally, using the definition (4.5*b*) for $k$, we find

$$k = 0.5\,Pr\,\omega^{-2}, \tag{7.15}$$

which is identical to (5.20*b*). This result confirms well the lack of effect of the thermal boundary conditions mentioned in §6.2.

## 8. Other non-dimensionalization and comparisons

We can remark that the asymptotic expression (7.15), for $\epsilon/\omega \ll 1$, can be expressed in terms of dimensional variables as

$$\eta_c = 0.5(\alpha\gamma)^2\,(r_0^4/\nu\kappa)\,[\epsilon^*/\omega^*]^2, \tag{8.1}$$

or

$$R_0^c/R_{00}^c = 1 + 0.5\,\alpha\gamma g_0^{-1}[\epsilon^*/\omega^*]^2, \tag{8.2}$$

where $[\epsilon^*/\omega^*]$ represents a vibrational velocity and $[g_0/\alpha\gamma]^{\frac{1}{2}}$ a reference velocity. The expression (8.2) is independent of $r_0, \nu, \kappa$ and $Pr$, while the expression (8.1) shows that $r_0^2/(\nu\kappa)^{\frac{1}{2}}$ should be take as a reference time, as in the study of Gershuni *et al.* (1970). Introducing the new non-dimensionalized frequency $\Omega = \omega/Pr^{\frac{1}{2}}$, which corresponds to this reference time, the asymptotic expression (7.15) becomes independent of $Pr$, and we have, respectively, in terms of $\epsilon$ and $\epsilon_v$ (with $\epsilon_v = \epsilon^*/g_0$)

$$\eta_c = 0.5[\epsilon/\Omega]^2 \tag{8.3a}$$

and

$$\eta_c = 0.5[R_{00}^c\,\epsilon_v/\Omega]^2. \tag{8.3b}$$

It would also be interesting, instead of $k$, to introduce the variable $k''$ defined by

$$k'' = k\Omega^2, \tag{8.4}$$

such that

$$\eta_c = k''[\epsilon/\Omega]^2, \tag{8.5a}$$

and

$$\eta_c = k''[R_{00}^c\,\epsilon_v/\Omega]^2. \tag{8.5b}$$

Then the limit when $\Omega \to \infty$ simply is a constant, $k'' = 0.5$, as illustrated in figure 5.

The relation (7.14) can be compared to the relation (4.4) of Gershuni *et al.* (1970) for large $\omega$; indeed, by putting $R = R_0^c/R_{00}^c$ in (4.7), we have

$$R - 1 = 0.5\,R^2 k_v\,\epsilon_v^2 \quad \text{with}\, k_v = R_{00}^c\,Pr\,\omega^{-2}. \tag{8.6}$$

The expression (4.4) of Gershuni *et al.* (1970) recovers (8.6) but with a different coefficient, $k_v = m^{-2}Pr\,\omega^{-2}$, where $m$ is such that $R_{00}^c\,m^2 = 1 + (\frac{1}{2}\mu_1^2 - 4)/(\mathrm{b}+3)^2$; $b$ being the Biot number as introduced in the condition (3.5*a*) and $\mu_1$ the first zero of $J_1(\mu_1)$, i.e. $\mu_1 = 3.832$. For the limiting case $b \to \infty$ (conducting walls), where $R_{00}^c\,m^2 = 1$, the two expressions for $k_v$ are identical. This perfect agreement is probably due to the fact that in this case all the matrices ($\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{A}_1, \boldsymbol{A}_2$ and $\boldsymbol{C}$) are diagonal and the first-order approximation of Gershuni *et al.* (1970) in their relation (1.17) is valid. But, for finite $b$, the expression (4.4) of Gershuni *et al.* (1970) would involve an extra effect of the boundary conditions, while in (8.6) this effect of the boundary conditions enters only through $R_{00}^c$; the first-order approximation used by these authors seems to be no longer valid for finite $b$.

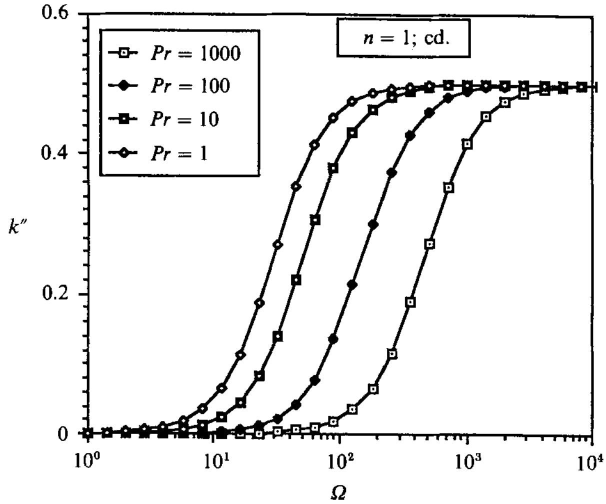


FIGURE 5. Modified alteration factor $k''$ *vs.* frequency $\Omega$, for the conducting case; $1 \leqslant Pr \leqslant 1000$.

We can remark that the expression (8.6), like the relation (4.4) of Gershuni *et al.* (1970), admits two solutions, one valid for $R_0^c$ close to $R_{00}^c$ and the other for large $R_0^c$. The first solution, which has to be retained when $\epsilon_v \ll \omega$, reads

$$R_1 = 1 + 0.5\, k_v\, \epsilon_v^2. \tag{8.7}$$

It corresponds, in fact, to $\eta_1$ given by (4.7*a*), that can be written as

$$\eta_1 = 0.5\, R_{00}^c\, k_v\, \epsilon_v^2 = 0.5\, Pr\, \omega^{-2}\, R_{00}^{c2}\, \epsilon_v^2. \tag{8.8}$$

Another interesting feature concerning the use of the frequency $\Omega$ is that, for conducting walls, the alteration factor $k''$ can be expressed from (6.17) as

$$k''(\Omega, Pr) = 0.5\,[1 + R_{00}^c (Pr^{\frac{1}{2}} + Pr^{-\frac{1}{2}})^2\, \Omega^{-2}]^{-1}. \tag{8.9}$$

This expression exhibits a 'symmetry' with respect to $Pr = 1$ (i.e. $k''(\Omega, Pr) = k''(\Omega, 1/Pr)$), for any values of $\Omega$, and presents a maximum at $Pr = 1$. The graphs of $k''$ are plotted as a function of $\Omega$ in figure 5, for several values of $Pr$ (namely 1, $10^{-1}$, $10^{-2}$ and $10^{-3}$).

For finite $\omega$, a direct comparison of the alteration factors (with the ones available in the literature) can only be made with the results concerning the modulated-Bénard problem with surface-temperature modulation which has been considered by Venezian (1969), Rosenblat & Herbert (1970) and Roppo *et al.* (1984). An analytic expression based on series expansions has been proposed by Venezian (1969) (see his expression (45)). The comparison has been made through the alteration parameter $R_{02}/R_{00}$ plotted by Roppo *et al.* (1984) against $\omega$, in the range $0 \leqslant \omega \leqslant 100$. This parameter is simply obtained from the Venezian's formula (45) by dividing it by the neutral stability threshold for free and conducting horizontal surfaces ($R_{00} = \frac{27}{4}\pi^4$). With our notation, the Venezian parameters $R_\sigma$ and $R_{02}/R_{00}$, respectively correspond to the ratio $\eta_1/R_{00}^c$ and to the product $kR_{00}^c$ (where $\eta_1$ and $k$ are give by (4.7*a*) and (6.17) respectively, and where $R_{00}^c = 215.56$ for conducting sidewalls). The curves giving $kR_{00}^c$ versus $\omega$ are plotted in figure 6 for three values of $Pr$, namely $Pr = 0.1$, 1 and 10; they show a perfect agreement for $\omega = 0$ between our results for a vertical cylinder of infinite length in the conducting case and the ones corresponding to

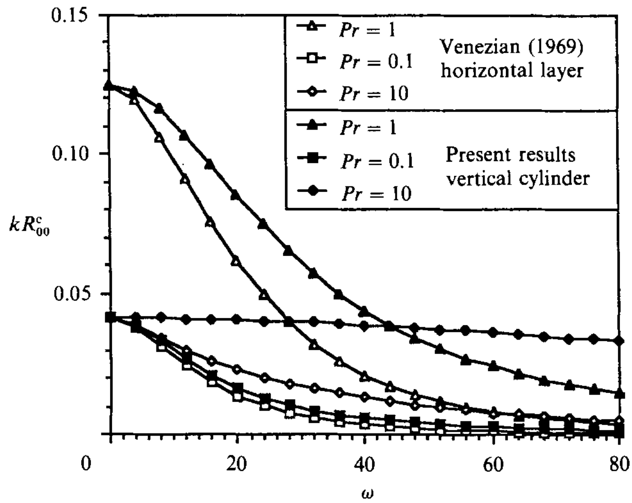


FIGURE 6. Modified alteration factor $kR^c_{00}$ *vs.* frequency, for the conducting case; ———, present results, vertical cylinder of infinite length (gravity modulation); ———, results plotted by Roppo *et al.* (1984) after Venezian (1969), horizontal layer of infinite extent (temperature modulation), for the modulated-Bénard problem, at $Pr = 0.1$, 1 and 10.

modulated-Bénard problems for free and conducting surfaces. Indeed, by using (4.7*a*) and (6.17), we get for small $\epsilon_v$

$$\eta_1/R^c_{00} = kR^c_{00}\,\epsilon^2_v, \tag{8.10}$$

with

$$kR^c_{00} = 0.5\,Pr\,R^c_{00}[\omega^2 + R^c_{00}(Pr+1)^2]^{-1} \tag{8.11a}$$

$$= 0.5\,[\omega^2(Pr\,R^c_{00})^{-1} + (Pr^{-\frac{1}{2}} + Pr^{\frac{1}{2}})^2]^{-1}, \tag{8.11b}$$

At $\omega = 0$, the expression (8.11*b*) gives

$$kR^c_{00} = 0.5\,(\mathrm{Pr}^{-\frac{1}{2}} + \mathrm{Pr}^{\frac{1}{2}})^{-2}, \tag{8.12}$$

which is strictly identical to the expressions obtained by the previous authors (relations (46) by Venezian 1969, and (4.16) by Rosenblat & Herbert 1970) for the modulated-Bénard problem with surface-temperature modulations. Such an identity proves that for $\omega \to 0$ the stability alteration (with respect to the unmodulated case) is not only independent of the type of periodic modulation (surface-temperature or gravity) as previously mentioned by Gershuni *et al.* (1970), but also independent of the type of geometry (confined or not confined). In the same manner the 'symmetry' property of (6.18) which associates the results at $Pr$ and $1/Pr$, appears to be quite 'universal'. It can be noted that Homsy (1974) also mentioned a similar property in the Bénard configuration according to the results of the energy method.

We can also observe in figure 6 a qualitative agreement, for $Pr \leqslant 1$, between our results and the ones corresponding to the modulated-Bénard problem, in the domain $0 < \omega \leqslant 100$. This agreement is confirmed by additional comparisons given in figure 7 (*a*), for $Pr = 0.01$ and $Pr = 0.001$; a quite similar behaviour of the alteration parameter is found for the cylinder and the Bénard problem on the range $0 < \omega \leqslant 20$. At $Pr = 10$, on the contrary, a quite different behaviour is observed; it is more accentuated again at $Pr = 100$, as shown in figure 7 (*b*). The expression (8.11*a*) shows that the results for the cylinder at large $Pr$ behaves like $\frac{1}{2}/(Pr+2)$ when $\omega \ll Pr$, while Venezian's expression (45) shows that the modulation parameter becomes negative

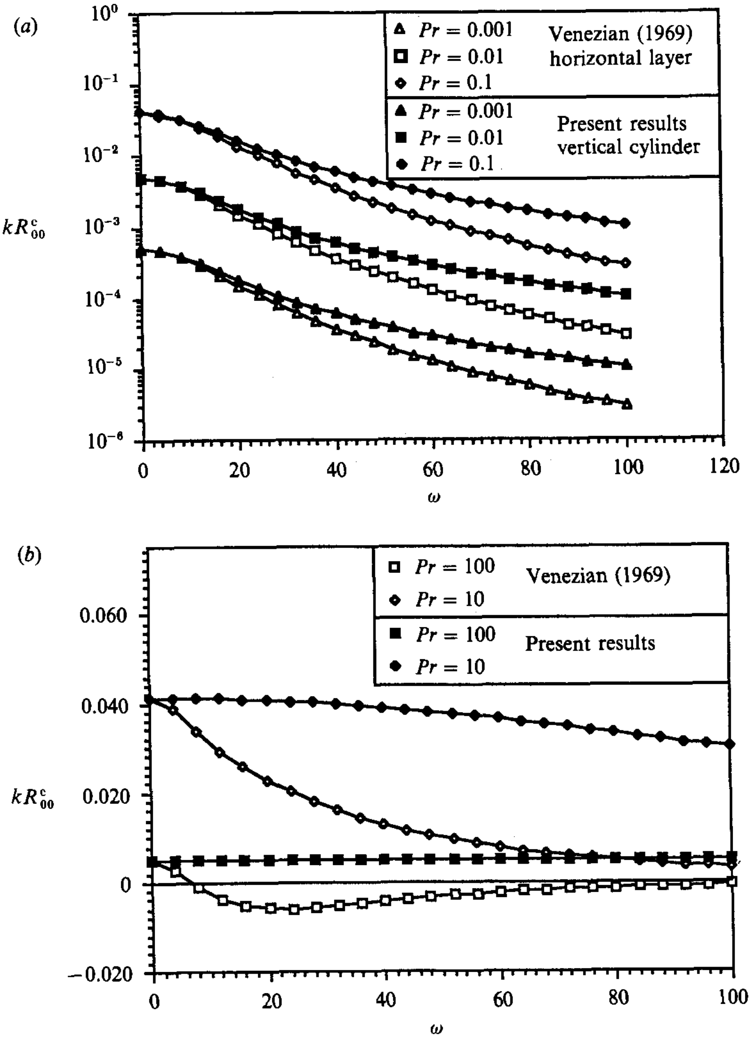


FIGURE 7. (*a*) Modified alteration factor $kR^c_{00}$ *vs.* frequency, for the conducting case; comparison with the results of Venezian (1969) for the modulated-Bénard problem, at $Pr = 0.001$, 0.01 and 0.1. (*b*) Modified alteration factor $kR^c_{00}$ *vs.* frequency, for the conducting case; comparison with the results of Venezian (1969) for the modulated-Bénard problem, at $Pr = 10$ and 100.

after $\omega = 7$ for $Pr = 100$. Thus, for high $Pr$, the constraint modulations can be destabilizing for the modulated-Bénard problem, while they are always stabilizing in the conducting case for the vertical cylinder.

## 9. Results and discussion

We have seen that a perfect agreement is found, in the case $n = 1$, between the values of $k$ given by the analytical methods presented in §5.1 and §7 and by the matrix method of §6 when using 30 and 5 trial functions for, respectively, insulated and conducting walls.

The knowledge of $k$, which is analytically given by (5.15) and (6.17), respectively, in the insulated and conducting cases, permits us to determine $\eta_1$ from (4.7*a*).

However we have seen that, for $k > 0$ the solution only exists for $\epsilon_v \leqslant \epsilon_v^0$, where $\epsilon_v^0 = 0.5\ [kR_{00}^c]^{-\frac{1}{2}}$. Values of $\epsilon_v^0$ can be simply derived in some particular cases, as follows. For conducting walls, we have seen through (6.17) that $k$ is maximum at $\omega = 0$ for any $Pr$. This maximum is given by (6.18); thus a lower limit of $\epsilon_v^0$ can be obtained as

$$\epsilon_v^0(0) = \sqrt{2} \quad (Pr^{-\frac{1}{2}} + Pr^{\frac{1}{2}}); \tag{9.1}$$

for example $\epsilon_v^0(0) = \sqrt{2}$ at $Pr = 1$. For large $\omega$, in both conducting and adiabatic cases, the use of (7.15) leads to

$$\epsilon_v^0 = \omega[2R_{00}^c\, Pr]^{-\frac{1}{2}}. \tag{9.2}$$

For conducting walls, the existence of a maximum enhancement for $\omega = 0$ and $Pr = 1$, as shown by (6.17), was also exhibited in previous studies concerning the Bénard problem with surface temperature modulation (see Venezian 1969). From a physical point of view, it is well known that, for very low frequencies, surface temperature modulations affect the entire volume of the fluid exactly as gravity modulations and then, the same law must describe the alteration of the stability limit for these two kinds of modulation. The new feature obtained herein is the absence of the effect of the geometry configuration (e.g. confined or not) on this alteration, as shown in §8 by a direct comparison with the results of Venezian (1969), Rosenblat & Herbert (1970) and Roppo *et al.* (1984), while the boundary conditions (conducting or insulated walls) have a significant effect as shown in figure 2.

For high frequencies, it is physically difficult to make direct comparisons with the Bénard problem with surface temperature modulation since, under such a modulation, only a thin layer near the walls is concerned and then, the equilibrium state tends to that of the unmodulated case. However, under gravity modulation, high frequencies correspond to a renormalization of the static gravity field, and then, when frequency modulations are large enough in comparison with the characteristic time for the diffusion processes (temperature and vorticity), the buoyancy force takes a mean value which leads to the equilibrium state of the unmodulated case. This is in agreement with the experiment of Donnelly (1964) concerning the stability of the flow between rotating cylinders where the inner cylinder has a modulated angular speed. He found that the increase of the frequency makes the effect of the modulations negligible.

To our knowledge there is no asymptotic relation for large $\omega$ in the previous works allowing, as here, one to conclude as to the similarity of the behaviour of systems under the modulation of surface temperature or gravity. Under the present assumptions (i.e. $\epsilon \ll \omega$), the alteration behaves as $\omega^{-2}$ for both kinds of modulation. For the gravity modulation, the effect of the thermal boundary conditions clearly appears; the alteration is simply proportional to $R_{00}^c$. This result shows a difference from the results of Gershuni *et al.* (1970), which show an extra effect of these thermal conditions.

For insulated walls, two domains of $Pr$ have been found ($0 < Pr < 8 \times 10^{-3}$ and $8 < Pr < \infty$), in which the effect of modulation is a destabilization one for small $\omega$. The existence of such a destabilization, due to the modulation, has already been mentioned for the modulated Bénard problem in infinite horizontal layers under special circumstances. Venezian (1969) reported such an effect in the case of free and conducting horizontal surfaces, when the temperature of these surfaces is modulated in phase. Homsy (1974), also for free and conducting horizontal surfaces, but with a gravity modulation, showed that $R_0^c < R_{00}^c$, in the extreme cases where $Pr \rightarrow 0$ and $Pr \rightarrow \infty$. In addition, identical values of $R_0^c$ were found by these authors in both these

| | |
|---|---|
| $g$-jitter frequency (Hz) | $10^{-1} \leqslant \omega^* \leqslant 10^2$ |
| $g$-jitter amplitude (cm s$^{-2}$) | $10^{-2} \leqslant \epsilon^* \leqslant 10^{-1}$ |
| Mean $\mu g$ (cm s$^{-2}$) | $10^{-2} \leqslant g_0 \leqslant 10^{-1}$ |
| Furnace radius (cm) | $1 \leqslant r_0 \leqslant 10$ |
| Temperature gradient (°/cm) | $1 \leqslant \gamma \leqslant 10$ |
| Time reference | $t_{\mathrm{ref}} = r_0^2/\kappa$ |
| Non-dimensional frequency | $\omega = \omega^*\, t_{\mathrm{ref}} = \omega^*\, r_0^2/\kappa$ |

TABLE 2. Typical microgravity conditions

extreme cases; a similar property can be seen in the present results for the adiabatic case, for $\omega \rightarrow 0$ (figure 2).

We also proved in §7 that the asymptotic relation (5.21) established for small $\epsilon$ is still valid for finite $\epsilon$ and large $\omega$, such that $\epsilon \ll \omega$ (or $\epsilon^* \ll \omega^*$). The relation (7.15) shows that the modulation always leads to a stabilization for large $\omega^*$, even for finite $\epsilon^*$. The inequality $\epsilon^* \ll \omega^*$, which limits the mathematical validity of the method presented herein, is not a severe physical limitation; in particular, it is easily satisfied for several classes of fluids (see tables 1 and 2), in the case of the $g$-jitter modulations.

## 10. Conclusion

The effect of the gravity modulation on the hydrodynamic instability (onset of convection) in a long vertical cylinder has been emphasized. A method has been proposed to analyse the stability threshold, $R_0^c$, in the case of periodic gravity modulation of small amplitude $\epsilon$, by using the Floquet theory. Attention has been focused on the alteration of this stability threshold compared to the one of the unmodulated case, $R_{00}^c$. These two stability thresholds being connected by the relation $R_0^c = R_{00}^c + k\,\epsilon^2$, only the value of the alteration factor $k$ has to be determined. In fact, as $\epsilon$ depends on $R_0^c$, another modulation amplitude $\epsilon_v = \epsilon^*/g_0 = \epsilon/R_0$ has also been used. We have shown that the expression giving $R_0^c$ in terms of $\epsilon_v$ is quadratic, and that only the solution corresponding to the smallest $R_0^c$ has to be retained; it is expressed as $R_0^c = R_{00}^c\,(1+k\,R_{00}^c\,\epsilon_v^2)$.

To determine $k$, an analytical approach is possible for certain cases, as presented in §5 for insulated walls. But for both conducting and insulated walls a quite general technique is described in §6; this matrix technique has been used for the azimuthal wavenumber $n = 1$, which corresponds to the most unstable situation for unmodulated cases in long cylinders.

The values of $k$ given by both analytical and matrix approaches are identical for all the cases considered herein, i.e. for a very wide range of $\omega$ and $Pr$ values. Some situations have been found where the gravity modulation has a destabilizing effect; this occurs for insulated walls and small $\omega$ in the ranges $0 < Pr < 8\times 10^{-3}$ and $8 < Pr < \infty$. In all the other cases, the gravity modulation is stabilizing.

In the conducting case, a completely new and very simple expression for $k$, valid for any $Pr$ and $\omega$, has been derived. This expression (6.17) shows that $k$ is always positive (stabilizing effect) and presents a maximum for $Pr = 1$ at $\omega \rightarrow 0$. It has been possible to compare the product $k\,R_{00}^c$ obtained in our case with previous results obtained by Venezian (1969), Rosenblat & Herbert (1970) and Roppo *et al.* (1984) for the modulated-Bénard problem (i.e. without lateral confinement). This alteration factor, $kR_{00}^{c}$, is shown to be strictly identical, in the limiting case $\omega = 0$, for the long vertical cylinder and the Bénard problems. The behaviour of $kR_{00}^{c}$ is still quite similar in the range $0 \leqslant \omega \leqslant 20$ for $Pr \leqslant 1$, but it changes completely for high $Pr$ where a destabilization occurs for small $\omega$ in the modulated-Bénard problem.

For large $\omega$, an asymptotic analysis has been done for small $\epsilon$ in §5.2; it has been extended in §7 to the case of finite $\epsilon$, for quite general boundary conditions and periodic constraints. A quite universal law $k = 0.5\,Pr\,\omega^{-2}$, valid for adiabatic and conducting conditions (in fact, for any value of the Biot number), has been found which shows a stabilizing effect of the gravity modulation for any $Pr$. This law agrees, in the conducting case, with the results obtained by Gershuni *et al.* (1970), but a fundamental difference is exhibited for finite value of the Biot number, $b$, where the solutions obtained by Gershuni *et al.* (1970) involve an additional effect of $b$.

The matrix approach will be extended to the case of a finite cylinder (finite length) in the near future.

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